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A NOTE ON THE SOLVABILITY OF THE DIOPHANTINE EQUATION

$$1^n + 2^n + \dots + m^n = G(m+1)^n$$

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A note on the solvability of the diophantine equation

$$1^n + 2^n + \dots + m^n = G(m+1)^n$$

by

J. van de Lune & H.J.J. te Riele

ABSTRACT

Concerning the diophantine equation

$$(*) \quad 1^n + 2^n + \dots + m^n = G(m+1)^n,$$

where G is a fixed positive rational, it is shown that the set of all $n \in \mathbb{N}$ for which $(*)$ has a solution $m \in \mathbb{N}$, has natural density zero, provided that $G > \frac{1}{e^{2\pi}-1}$. As a consequence one has that the diophantine equation

$$1^n + 2^n + \dots + m^n = (m+1)^n$$

is almost never solvable.

KEY WORDS & PHRASES: *Sums of powers of integers, diophantine equation, uniform distribution.*

THEOREM 1. If $G \in \mathbb{Q}^+$ and $n \in \mathbb{N}$ are given then there exists at most one $m \in \mathbb{N}$ such that

$$(1) \quad 1^n + 2^n + \dots + m^n = G(m+1)^n.$$

PROOF. Writing

$$(2) \quad \sigma_m(n) = \sum_{k=1}^m k^n, \quad (m, n \in \mathbb{N})$$

we have (cf. [3; p.5])

$$(3) \quad \frac{m^{n+1}(m+1)^n}{(m+1)^{n+1} - m^{n+1}} < \sigma_m(n) < \frac{m^n(m+1)^{n+1}}{(m+1)^{n+1} - m^{n+1}}.$$

Hence, defining $\Theta = \Theta(m, n)$ by

$$(4) \quad \sigma_m(n) = \frac{m^n(m+1)^{n(m+\Theta)}}{(m+1)^{n+1} - m^{n+1}}$$

we have

$$(5) \quad 0 < \Theta(m, n) < 1, \quad \forall m, n \in \mathbb{N}.$$

It follows that if (1) is solvable we must have

$$(6) \quad \sigma_m(n) = G(m+1)^n$$

so that (writing Θ instead of $\Theta(m, n)$)

$$(7) \quad \frac{m^n(m+1)^{n(m+\Theta)}}{(m+1)^{n+1} - m^{n+1}} = G(m+1)^n$$

or, equivalently,

$$(8) \quad m^{n+1} + \Theta m^n = G(m+1)^{n+1} - G m^{n+1}.$$

Since $\Theta > 0$ we obtain

$$(9) \quad (G+1)m^{n+1} < G(m+1)^{n+1}$$

which may also be written as

$$(10) \quad m < -1 + \frac{1}{1-B} \frac{1}{1/(n+1)}$$

where

$$(11) \quad B \stackrel{\text{def}}{=} \frac{G}{G+1}.$$

If (1) is solvable we also have

$$(12) \quad \sigma_{m+1}(n) = (G+1)(m+1)^n$$

so that (writing θ instead of $\theta(m+1, n)$)

$$(13) \quad \frac{(m+1)^n (m+2)^n (m+1+\theta)}{(m+2)^{n+1} - (m+1)^{n+1}} = (G+1)(m+1)^n$$

which may also be written as

$$(14) \quad (m+2)^{n+1} + (\theta-1)(m+2)^n = (G+1)\{(m+2)^{n+1} - (m+1)^{n+1}\}.$$

Since $\theta < 1$ it follows that

$$(15) \quad (G+1)(m+1)^{n+1} > G(m+2)^{n+1}$$

from which it is easily seen that

$$(16) \quad m > -2 + \frac{1}{1-B} \frac{1}{1/(n+1)}$$

where B is as in (11).

Combining (10) and (16) we obtain

$$(17) \quad -2 + \frac{1}{1-B} \frac{1}{1/(n+1)} < m < -1 + \frac{1}{1-B} \frac{1}{1/(n+1)}$$

completing the proof of theorem 1. \square

From the above proof it is clear that if $G \in \mathbb{Q}^+$ and $n \in \mathbb{N}$ are such that (1) is solvable for m , then this solution is given by

$$(18) \quad m = m(n) = [\lambda(n)]$$

where $\lambda(n)$ is defined (for all $n \in \mathbb{N}$) by

$$(19) \quad \lambda(n) = -1 + \frac{1}{1-B^{1/(n+1)}}$$

and $[x]$ denotes the greatest integer not exceeding x .

Now observe that

$$\begin{aligned} (20) \quad \lambda(n) &= -1 + \frac{1}{1-B^{1/(n+1)}} = -1 - \frac{1}{B^{1/(n+1)} - 1} = -1 - \frac{1}{\exp(\frac{1}{n+1} \log B) - 1} = \\ &= -1 - \left\{ \frac{1}{\frac{1}{n+1} \log B} - \frac{1}{2} + O\left(\frac{1}{n}\right) \right\} = \frac{n+1}{\log \frac{1}{B}} - \frac{1}{2} + O\left(\frac{1}{n}\right), \quad (n \rightarrow \infty). \end{aligned}$$

Since e^r is irrational for every positive rational r it follows that $\log \frac{1}{B} = \log(1 + \frac{1}{G})$ is irrational. Hence, (see [2; p.92, Satz 9]) the sequence

$\left\{ (n+1) \left(\log \frac{1}{B} \right)^{-1} \right\}_{n=1}^{\infty}$ is uniformly distributed modulo 1 (u.d. mod 1).

Now recall that if a real sequence $\{\alpha(n)\}_{n=1}^{\infty}$ is u.d. mod 1 and if $\{\beta(n)\}_{n=1}^{\infty}$ is some *convergent* real sequence then also $\{\alpha(n) + \beta(n)\}_{n=1}^{\infty}$ is u.d. mod 1.

From these observations it follows that $\{\lambda(n)\}_{n=1}^{\infty}$ is u.d. mod 1.

Since $m(n)$, as defined by (18), is an integer and

$$(21) \quad 0 < \lambda(n) - m(n) < 1$$

it follows that $\{\lambda(n) - m(n)\}_{n=1}^{\infty}$ is uniformly distributed on the interval $(0,1)$.

Now fix any $G \in \mathbb{Q}$ such that

$$(22) \quad G > \frac{1}{e^{2\pi} - 1}$$

and let S be the set of all $n \in \mathbb{N}$ for which (1) is solvable with respect to m . Then we have the following

THEOREM 2. *The set S has natural density zero.*

PROOF. If S is finite (possibly empty) then we are done.

If S contains infinitely many elements, it follows from (18), (20) and the definition of B that

$$(23) \quad \lim_{\substack{n \rightarrow \infty \\ n \in S}} \frac{n}{m(n)} = \log \frac{1}{B} = \log(1 + \frac{1}{G}).$$

For every $n \in S$ we have

$$(24) \quad \sigma_m(n) = G(m+1)^n, \quad (m = m(n))$$

so that, in view of (4), we have (writing Θ instead of $\Theta(m, n)$)

$$(25) \quad \frac{m^{n(m+1)} n^{(m+\Theta)}}{(m+1)^{n+1} m^{n+1}} = G(m+1)^n$$

or, equivalently,

$$(26) \quad m = \frac{1}{(\frac{G+1}{G} + \frac{\Theta}{mG})^{1/(n+1)} - 1}.$$

Next we investigate the asymptotic behaviour of $\{\lambda(n) - m(n)\}_{n \in S}$ as $n \rightarrow \infty$. Since for $n \in S$, $n \rightarrow \infty$, we have

$$\begin{aligned} (27) \quad \lambda(n) - m(n) &= \frac{n+1}{\log(1 + \frac{1}{G})} - \frac{1}{2} + o(\frac{1}{n}) + \\ &- \frac{1}{\exp\{\frac{1}{n+1} \log(\frac{G+1}{G} + \frac{\Theta}{mG})\} - 1} = \frac{n+1}{\log(1 + \frac{1}{G})} - \frac{1}{2} + o(\frac{1}{n}) + \\ &- \left\{ \frac{1}{\frac{1}{n+1} \log(\frac{G+1}{G} + \frac{\Theta}{mG})} - \frac{1}{2} + o(\frac{1}{n}) \right\} = \\ &= (n+1) \left\{ \frac{1}{\log \frac{G+1}{G}} - \frac{1}{\log(\frac{G+1}{G} + \frac{\Theta}{mG})} \right\} + o(\frac{1}{n}) = \\ &= (n+1) \frac{\log(1 + \frac{\Theta}{m(G+1)})}{\log \frac{G+1}{G} \log(\frac{G+1}{G} + \frac{\Theta}{mG})} + o(\frac{1}{n}) = \\ &= \left\{ \frac{1}{(\log \frac{G+1}{G})^2} + o(1) \right\} (n+1) \left\{ \frac{\Theta}{m(G+1)} + o(\frac{1}{m}) \right\} + o(\frac{1}{n}) = \\ &= \frac{1}{(\log \frac{G+1}{G})^2} \frac{(n+1)\Theta(m, n)}{m(G+1)} + o(1), \end{aligned}$$

it follows that $\{\lambda(n) - m(n)\}_{n \in S}$ is *convergent* as soon as $\Theta(m(n), n)$ converges for $n \rightarrow \infty$.

For the moment let us assume that $\{\Theta(m(n), n)\}_{n=1}^{\infty}$ is convergent. Then $\{\lambda(n) - m(n)\}_{n \in S}$ is a convergent subsequence of $\{\lambda(n) - m(n)\}_{n=1}^{\infty}$, the latter sequence being uniformly distributed on $(0, 1)$. From this it follows that S has natural density zero. Hence, in order to complete the proof of theorem 2 it suffices to prove the following

THEOREM 3. *For any $n \in \mathbb{N}$ let $m(n)$ be defined by (18). Then*

$$(28) \quad \lim_{n \rightarrow \infty} \Theta(m(n), n)$$

exists and is equal to

$$(29) \quad G+1 - G(G+1) \log\left(1 + \frac{1}{G}\right)$$

provided that $G > \frac{1}{e^{2\pi}-1}$.

PROOF. Similarly as in [4] we consider the asymptotic behaviour of the sums $\sigma_m(n)$ as $n \rightarrow \infty$ and $\frac{n}{m} \rightarrow \alpha = \log\left(1 + \frac{1}{G}\right)$.

By means of the Euler-Maclaurin summation formula one may show that

$$(30) \quad \frac{\sigma_m(n)}{m} = \frac{m}{n+1} + \frac{1}{2} + \sum_{r=1}^{\left[\frac{n}{2}\right]} \frac{B_{2r}}{2r} \left(\frac{n}{2r-1}\right)_m^{-2r+1}$$

where the Bernoulli numbers B_r are defined by

$$(31) \quad \frac{z}{e^z - 1} = \sum_{r=0}^{\infty} \frac{B_r}{r!} z^r, \quad (|z| < 2\pi).$$

It is well known that for any real $\alpha \neq 0$ (see [1; p.528])

$$(32) \quad \frac{1}{e^\alpha - 1} = \frac{1}{\alpha} - \frac{1}{2} + \sum_{r=1}^k \frac{B_{2r}}{(2r)!} \alpha^{2r-1} + R_k(\alpha)$$

where

$$(33) \quad R_k(\alpha) = \frac{\alpha^{2k+1}}{e^\alpha - 1} \int_0^1 P_{2k+1}(x) e^{\alpha x} dx$$

so that

$$(34) \quad \frac{1}{e^{n/m} - 1} - \frac{m}{n} + \frac{1}{2} = \sum_{r=1}^k \frac{B_{2r}}{(2r)!} \left(\frac{n}{m}\right)^{2r-1} + R_k\left(\frac{n}{m}\right).$$

Taking $k = [\frac{n}{2}]$ in (34) it follows from (30) and (34) that

$$(35) \quad \left\{ \frac{1}{e^{n/m} - 1} - \frac{m}{n} + \frac{1}{2} \right\} - \left\{ \frac{\sigma_m^{(n)}}{m^n} - \frac{m}{n+1} - \frac{1}{2} \right\} =$$

$$= \frac{1}{e^{n/m} - 1} + 1 - \frac{m}{n(n+1)} - \frac{\sigma_m^{(n)}}{m^n} =$$

$$= \sum_{r=2}^k \frac{B_{2r}}{(2r)!} \left(\frac{n}{m}\right)^{2r-1} \left\{ 1 - \frac{n(n-1)\dots(n-2r+2)}{n^{2r-1}} \right\} + R_k\left(\frac{n}{m}\right) =$$

$$= \sum_{r=2}^k \frac{B_{2r}}{(2r)!} \left(\frac{n}{m}\right)^{2r-1} \delta_n(2r-2) + R_k\left(\frac{n}{m}\right)$$

where $\delta_n(\cdot)$ is defined by

$$(36) \quad \delta_n(a) = 1 - \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\dots\left(1 - \frac{a}{n}\right), \quad (a \in \mathbb{N}).$$

From the definition of $\delta_n(a)$ it is easily seen that for any fixed $a \in \mathbb{N}$

$$(37) \quad \lim_{n \rightarrow \infty} n \delta_n(a) = 1 + 2 + \dots + a = \frac{1}{2} a(a+1).$$

By mathematical induction one may show that

$$(38) \quad (0 <) \delta_n(a) \leq \frac{a(a+1)}{2n}, \quad (1 \leq a < n; n \geq 2).$$

As a consequence we have

$$(39) \quad \left| \frac{B_{2r}}{(2r)!} \left(\frac{n}{m}\right)^{2r-1} \delta_n(2r-2) \right| \leq \frac{|B_{2r}|}{(2r)!} \left(\frac{n}{m}\right)^{2r-1} \frac{(2r-2)(2r-1)}{2n}$$

so that in view of the fact that

$$(40) \quad \lim_{n \rightarrow \infty} \frac{n}{m(n)} = \log\left(1 + \frac{1}{G}\right)$$

and our assumption that

$$(41) \quad G > \frac{1}{e^{2\pi-1}}, \quad (\text{so that } \log(1 + \frac{1}{G}) < 2\pi)$$

we have that for some $\varepsilon > 0$

$$(42) \quad \left| \frac{B_{2r}}{(2r)!} \left(\frac{n}{m(n)}\right)^{2r-1} n \delta_n(2r-2) \right| \leq \frac{1}{2} \frac{|B_{2r}|}{(2r)!} (2\pi-\varepsilon)^{2r-1} (2r-2)(2r-1)$$

if n is sufficiently large (independent of r).

The right hand side of (42) is the general term of a convergent series (compare (31)) so that, by a uniform convergence argument (or by Lebesgue's dominated convergence theorem) we obtain (as before we take $k = [\frac{n}{2}]$)

$$\begin{aligned} (43) \quad \lim_{n \rightarrow \infty} \sum_{r=2}^k \frac{B_{2r}}{(2r)!} \left(\frac{n}{m}\right)^{2r-1} n \delta_n(2r-2) &= \\ &= \sum_{r=2}^{\infty} \frac{B_{2r}}{(2r)!} \left(\log(1 + \frac{1}{G})\right)^{2r-1} \frac{1}{2} (2r-1)(2r-2) = \\ &= \frac{1}{2} \left(\log(1 + \frac{1}{G})\right)^2 \sum_{r=2}^{\infty} \frac{B_{2r}}{(2r)!} (2r-1)(2r-2) \left(\log(1 + \frac{1}{G})\right)^{2r-3} = \\ &= \frac{1}{2} \left(\log(1 + \frac{1}{G})\right)^2 \frac{d^2}{dx^2} \left\{ \sum_{r=2}^{\infty} \frac{B_{2r}}{2r!} x^{2r-1} \right\}_{x=\log(1 + \frac{1}{G})}. \end{aligned}$$

Now observe that (see [1; p.204])

$$(44) \quad x \cot x = 1 - \frac{B_2}{2!} (2x)^2 + \frac{B_4}{4!} (2x)^4 - + \dots$$

from which it is easily seen that

$$(45) \quad \sum_{r=2}^{\infty} \frac{B_{2r}}{(2r)!} x^{2r-1} = \frac{i}{2} \cot \frac{xi}{2} - \frac{1}{x} - \frac{x}{12}$$

so that

$$(46) \quad \frac{d^2}{dx^2} \left\{ \sum_{r=2}^{\infty} \frac{B_{2r}}{(2r)!} x^{2r-1} \right\} = -\frac{2}{x^3} - \frac{e^{-x}-e^x}{(e^{-x/2}-e^{x/2})^4}$$

which, for $x = \log(1 + \frac{1}{G})$ assumes the value

$$(47) \quad - \frac{2}{(\log(1 + \frac{1}{G}))^3} + G(G+1)(2G+1).$$

Hence, defining

$$(48) \quad \rho(n) = n \sum_{r=2}^k \frac{B_{2r}}{(2r)!} \left(\frac{n}{m}\right)^{2r-1} \delta_n(2r-2)$$

it follows from (35) that

$$(49) \quad \frac{\sigma_m(n)}{m^n} = \frac{1}{e^{n/m-1}} + 1 - \frac{m}{n(n+1)} - \frac{\rho(n)}{n} - R_k\left(\frac{n}{m}\right)$$

where, in view of (43), (46) and (47)

$$(50) \quad \lim_{n \rightarrow \infty} \rho(n) = \frac{1}{2}(\log(1 + \frac{1}{G}))^2 \left\{ - \frac{2}{(\log(1 + \frac{1}{G}))^3} + G(G+1)(2G+1) \right\} =$$

$$= - \frac{1}{\log(1 + \frac{1}{G})} + \frac{1}{2} G(G+1)(2G+1)(\log(1 + \frac{1}{G}))^2.$$

For $R_k(\frac{n}{m})$ we have the following estimate

$$(51) \quad |R_k(\frac{n}{m})| \leq \frac{(\frac{n}{m})^{2k+1}}{e^{n/m-1}} \int_0^1 |P_{2k+1}(x)| e^{\frac{nx}{m}} dx \leq \frac{(\frac{n}{m})^{2k+1}}{e^{n/m-1}} \cdot \frac{4}{(2\pi)^{2k+1}} e^{n/m} =$$

$$= \frac{4}{\pi} \frac{n/m}{e^{n/m-1}} e^{n/m} \left(\frac{n}{2\pi m}\right)^{2k} \leq C \left(\frac{n}{2\pi m}\right)^{2k}.$$

Since $\frac{n}{m} \rightarrow \log(1 + \frac{1}{G})$ as $n \rightarrow \infty$ and $G > \frac{1}{e^{2\pi}-1}$ (so that $\log(1 + \frac{1}{G}) < 2\pi$) it follows that $R_k(\frac{n}{m})$ tends exponentially fast to zero as $n \rightarrow \infty$.

As a simple consequence of (49), (50) and (51) we have

$$(52) \quad \lim_{n \rightarrow \infty} \frac{\sigma_m(n)}{m^n} = \frac{1}{e^{\log(1 + \frac{1}{G}) - 1}} + 1 = G + 1$$

so that

$$(53) \quad \lim_{n \rightarrow \infty} \frac{\sigma_m(n)}{m^n} \left\{ 1 - \left(1 - \frac{1}{m+1}\right)^{n+1} \right\} = (G+1) \left\{ 1 - e^{-\log(1 + \frac{1}{G})} \right\} = 1.$$

Now observe that

$$(54) \quad \Theta(m, n) = m \left\{ \frac{\sigma_m(n)}{m^n} \left(1 - \left(1 - \frac{1}{m+1} \right)^{n+1} \right) - 1 \right\} + \frac{\sigma_m(n)}{m^n} \left(1 - \left(1 - \frac{1}{m+1} \right)^{n+1} \right).$$

Hence in order to prove that $\lim_{n \rightarrow \infty} \Theta(m, n)$ exists we only need to study the asymptotic behaviour of

$$(55) \quad \begin{aligned} & m \left\{ \frac{\sigma_m(n)}{m^n} \left(1 - \left(1 - \frac{1}{m+1} \right)^{n+1} \right) - 1 \right\} = \\ & = m \left\{ \left(\frac{1}{e^{n/m-1}} - \frac{m}{n(n+1)} + 1 - \frac{\rho(n)}{n} - R \right) \left(1 - \left(1 - \frac{1}{m+1} \right)^{n+1} \right) - 1 \right\} = \\ & = -m \left(\frac{m}{n(n+1)} + \frac{\rho(n)}{n} \right) \left\{ 1 - \left(1 - \frac{1}{m+1} \right)^{n+1} \right\} + \\ & + m \left\{ \left(\frac{1}{e^{n/m-1}} + 1 \right) \left(1 - \left(1 - \frac{1}{m+1} \right)^{n+1} \right) - 1 \right\} - m R_k \left(\frac{n}{m} \right) \left\{ 1 - \left(1 - \frac{1}{m+1} \right)^{n+1} \right\}. \end{aligned}$$

Since $R_k \left(\frac{n}{m} \right)$ tends exponentially fast to zero and $m = O(n)$ it follows that

$$(56) \quad \lim_{n \rightarrow \infty} m R_k \left(\frac{n}{m} \right) \left\{ 1 - \left(1 - \frac{1}{m+1} \right)^{n+1} \right\} = 0.$$

Next we have

$$(57) \quad \begin{aligned} & \lim_{n \rightarrow \infty} m \left\{ \frac{m}{n(n+1)} + \frac{\rho(n)}{n} \right\} = \lim_{n \rightarrow \infty} \left\{ \frac{m^2}{n(n+1)} + \frac{m}{n} \rho(n) \right\} = \\ & = \frac{1}{(\log(1 + \frac{1}{G}))^2} + \frac{1}{\log(1 + \frac{1}{G})} \cdot \left\{ -\frac{1}{\log(1 + \frac{1}{G})} + \frac{1}{2} G(G+1)(2G+1) \left(\log(1 + \frac{1}{G}) \right)^2 \right\} \\ & = \frac{1}{2} G(G+1)(2G+1) \log(1 + \frac{1}{G}). \end{aligned}$$

Finally we have

$$\begin{aligned}
(58) \quad & m \left\{ \left(\frac{1}{e^{n/m-1}} + 1 \right) \left(1 - \left(1 - \frac{1}{m+1} \right)^{n+1} \right) - 1 \right\} = \\
& = m \left\{ \frac{e^{n/m}}{e^{n/m-1}} \left(1 - \left(1 - \frac{1}{m+1} \right)^{n+1} \right) - 1 \right\} = m \frac{e^{n/m} - e^{n/m} \left(1 - \frac{1}{m+1} \right)^{n+1} - e^{n/m+1}}{e^{n/m-1}} = \\
& = m \frac{1 - e^{n/m} \left(1 - \frac{1}{m+1} \right)^{n+1}}{e^{n/m-1}} = \frac{m}{e^{n/m-1}} \left\{ 1 - \exp \left(\frac{n}{m} + (n+1) \log \left(1 - \frac{1}{m+1} \right) \right) \right\} = \\
& = - \frac{m}{e^{n/m-1}} \frac{\exp \left(\frac{n}{m} + (n+1) \log \left(1 - \frac{1}{m+1} \right) \right) - 1}{(0 \neq) \frac{n}{m} + (n+1) \log \left(1 - \frac{1}{m+1} \right)} \cdot \left\{ \frac{n}{m} + (n+1) \log \left(1 - \frac{1}{m+1} \right) \right\}.
\end{aligned}$$

Observing that

$$(59) \quad \lim_{n \rightarrow \infty} \left\{ \frac{n}{m} + (n+1) \log \left(1 - \frac{1}{m+1} \right) \right\} = \log \left(1 + \frac{1}{G} \right) + \log e^{-\log \left(1 + \frac{1}{G} \right)} = 0$$

it follows that

$$\begin{aligned}
(60) \quad & \lim_{n \rightarrow \infty} (58) = \\
& = - \lim_{n \rightarrow \infty} \frac{m}{e^{n/m-1}} \left\{ \frac{n}{m} + (n+1) \log \left(1 - \frac{1}{m+1} \right) \right\} = \\
& = - \lim_{n \rightarrow \infty} \frac{m}{e^{n/m-1}} \left\{ \frac{n}{m} - (n+1) \left(\frac{1}{m+1} + \frac{1}{2(m+1)^2} + O\left(\frac{1}{n}\right) \right) \right\} = \\
& = - \lim_{n \rightarrow \infty} \frac{m}{e^{n/m-1}} \left\{ \frac{n}{m} - \frac{n+1}{m+1} - \frac{n+1}{2(m+1)^2} + O\left(\frac{1}{n}\right) \right\} = \\
& = - \lim_{n \rightarrow \infty} \frac{1}{e^{n/m-1}} \left\{ \frac{n-m}{m+1} - \frac{m(n+1)}{2(m+1)^2} \right\} = \\
& = - \frac{1}{\log \left(1 + \frac{1}{G} \right) - 1} \left\{ \log \left(1 + \frac{1}{G} \right) - 1 - \frac{1}{2} \log \left(1 + \frac{1}{G} \right) \right\} = \\
& = G \left\{ 1 - \frac{1}{2} \log \left(1 + \frac{1}{G} \right) \right\}.
\end{aligned}$$

Combining (53), (54), (55), (56), (57), (58) and (60) it follows that

$$(61) \quad \lim_{n \rightarrow \infty} \Theta(m(n), n) = 1 + G - G(G+1) \log\left(1 + \frac{1}{G}\right)$$

completing the proof of theorem 3 and hence that of theorem 2. \square

REFERENCES

- [1] KNOPP, K., *Theory and application of infinite series*, Blackie & Son Ltd. (1928).
- [2] KOKSMA, J.F., *Diophantische approximationen*, Springer (1936).
- [3] van de LUNE, J., *Monotonic approximation of integrals in relation to some inequalities for sums of powers of integers*, Mathematical Centre, Amsterdam, Report ZW 39/75.
- [4] van de LUNE, J. & H.J.J. te RIELE, *On a conjecture of Erdős (II)*, Mathematical Centre, Amsterdam, Report ZW 56/75.